QUATERNION ORDERS AND TERNARY QUADRATIC FORMS

STEFAN LEMURELL

Introduction

The main purpose of this paper is to provide an introduction to the arithmetic theory of quaternion algebras. However, it also contains some new results, most notably in Section 5. We will emphasise on the connection between quaternion algebras and quadratic forms. This connection will provide us with an efficient tool to consider arbitrary orders instead of having to restrict to special classes of them. The existing results are mostly restricted to special classes of orders, most notably to so called Eichler orders.

The paper is organised as follows. Some notations and background are provided in Section 1, especially on the theory of quadratic forms. Section 2 contains the basic theory of quaternion algebras. Moreover at the end of that section, we give a quite general solution to the problem of representing a quaternion algebra with given discriminant. Such a general description seems to be lacking in the literature.

Section 3 gives the basic definitions concerning orders in quaternion algebras. In Section 4, we prove an important correspondence between ternary quadratic forms and quaternion orders. Section 5 deals with orders in quaternion algebras over $\mathfrak{p}$-adic fields. The major part is an investigation of the isomorphism classes in the non-dyadic and 2-adic cases. The starting-point is the correspondence with ternary quadratic forms and known classifications of such forms. From this, we derive representatives of the isomorphism classes of quaternion orders. These new results are complements to existing more ring-theoretic descriptions of orders. In particular, they are useful for computations.

Finally, section 6 contains the basic theory of orders in quaternion algebras over algebraic number fields and the connection with the $\mathfrak{p}$-adic case. At the end of that section, we give an explicit basis of a maximal order when the discriminant of the algebra is a principal ideal. It is related to the results at the end of Section 2 and is more general than any other similar description known to the author.

1. Notation and some background

For convenience of the reader, we will recall the basic definitions of the theory of quadratic forms. Let K be a field with $\mathrm{char} K \neq 2$ and R any

subring of K containing 1. To every quadratic form

$$f = f(X_1, \dots, X_n) = \sum_{1 \le i \le j \le n} a_{ij} X_i X_j, \tag{1.1}$$

with coefficients $a_{ij} \in K$, we can associate a symmetric $n \times n$-matrix $M_f = (m_{ij})$, where

$$m_{ij} = \begin{cases} a_{ij}, & i < j \\ 2a_{ii}, & i = j \\ a_{ji}, & i > j, \end{cases}$$

and the corresponding bilinear form $B_f(x, y) = x^t M_f y$. A direct inspection gives $f(x) = \frac{1}{2} B_f(x, x)$. This is where we get into trouble, if $\mathrm{char} K = 2$. We will only consider non-degenerate forms, that is, forms for which $\det(M_f) \neq 0$. The form f is said to represent $a \in K$ over R, if there exists $x \in R^n$ such that $f(x) = a$. It is called isotropic, if there is a non-trivial representation of 0, otherwise it is called anisotropic. We will use the standard notation $f = \langle a_1 \rangle \perp \cdots \perp \langle a_n \rangle$ to denote the diagonal form $f = a_1 X_1^2 + \cdots + a_n X_n^2$.

Two forms f and g are called isometric over R, $f \cong g$, if there exists $T \in GL_n(R)$ such that $M_f = T^t M_g T$. They are said to be similar over R, denoted by $f \sim g$, if there exists an $u \in R^*$ such that $u \cdot f$ is isometric to g. Both isometry and similarity are equivalence relations, similarity obviously being coarser.

A form f like (1.1) is called integral over R if $a_{ij} \in R$. It is called primitive, if the ideal generated by the coefficients a_{ij} is equal to R.

The discriminant, $d(f)$, of a non-degenerate quadratic form f is defined to be the class of $\det(M_f)$ in $K^*/(K^*)^2$. The reason for taking classes modulo $(K^*)^2$ is to make it an invariant of isometry classes. Note that it is only an invariant of similarity classes in even dimensions, since multiplication by u multiplies the determinant by u^n. If the form f is integral over R, then the discriminant of f regarded as a form over R is the class of $\det(M_f)$ as an element in the multiplicative set $R \setminus \{0\}$ modulo $(R^*)^2$. In the case of quadratic forms of odd dimensions, it is customary and natural to take the discriminant to be the class of $\frac{1}{2}\det(M_f)$ instead, and we will follow this convention.

Let K be an algebraic number field with ring of integers R. This will be the case for much of the paper and exceptions will be clearly notified. Let Ω be the set of places (normalised valuations) on K, Ω_f the finite (non-archimedean) and Ω_∞ the infinite (archimedean) ones. If $\nu \in \Omega$, then K_ν will denote the completion of K with respect to ν, and if $\nu \in \Omega_f$, then R_ν will be the ring of integers in K_ν.

When a ring A is understood, then $\langle x_1, \dots, x_n \rangle$ will denote the free A-module generated by $\{x_1, \dots, x_n\}$.

2. Quaternion algebras

In this section, we will give the definition and some fundamental properties of quaternion algebras. We will reduce in generality along the way to make the exposition as simple and clear as possible.

(2.1) Definition. Let K be an arbitrary field. A quaternion algebra $\mathfrak{A}$ over K is a simple, central algebra of dimension 4 over K.

From Wedderburn's structure theorem on simple algebras [16, 2.5], one deduces that either $\mathfrak{A} \cong M_2(K)$ or $\mathfrak{A} \cong D$, where D is a division algebra with centre K.

We will from now on assume that char$K \neq 2$. With this condition, it is always possible to find a convenient 'diagonal' basis $1, i, j, ij$ of $\mathfrak{A}$ over K, which satisfies

$$i^2 = a,\ j^2 = b \text{ and } ij = -ji,\ \text{where } a, b \in K,\ ab \neq 0. \tag{2.2}$$

A proof of this can be found in a more general setting in [16, §17]. We will denote such an algebra by $(a, b)_K$.

(2.3) Remark. To make everything completely explicit, we remark that it is possible to embed $(a, b)_K$ in $M_2(K(\sqrt{a}))$ for example by

$$i \longmapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix},\ j \longmapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}.$$

From this it is clear that if a is a square in K, then $(a, b)_K \cong M_2(K)$. A necessary and sufficient condition for $(a, b)_K \cong M_2(K)$ is that a is the norm of an element in $K(\sqrt{b})$ with respect to K [16, 17.4]. Of course, one may interchange a and b in this remark. □

There is a natural involution in $\mathfrak{A}$, which in a basis satisfying (2.2) is given by

$$x = x_0 + x_1 i + x_2 j + x_3 ij \longmapsto \bar{x} = x_0 - x_1 i - x_2 j - x_3 ij.$$

One defines the (reduced) trace and (reduced) norm from $\mathfrak{A}$ into K by

$$Tr(x) = x + \bar{x} \text{ and } N(x) = x\bar{x}.$$

The norm is a quaternary quadratic form over K, with corresponding symmetric bilinear form given by $B(x, y) = Tr(x\bar{y})$. A direct calculation gives

$$N(x) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2,$$

if $x = x_0 + x_1 i + x_2 j + x_3 ij \in (a, b)_K$. From this, we see that the determinant of the norm on $(a, b)_K$ is equal to $16a^2b^2$. Hence, the discriminant of the norm form of a quaternion algebra is equal to 1.

Now define the set of pure quaternions $\mathfrak{A}_0$ to be

$$\mathfrak{A}_0 = \{q \in \mathfrak{A} : Tr(q) = 0\},$$

and denote the norm N restricted to $\mathfrak{A}_0$ by N_0. If we have chosen a basis of $\mathfrak{A}$ satisfying (2.2), then obviously $\mathfrak{A}_0 = \langle i, j, ij \rangle$. We conclude that $\mathfrak{A}_0$ is a

3-dimensional K-vector space and N_0 a ternary quadratic form. A nontrivial and interesting fact is that N is isotropic iff N_0 is isotropic [23, 42:12].

The norm form determines the quaternion algebra in the following sense [23, §57]:

(2.4) Proposition. *Let K be a field with char$K \neq 2$ and let $\mathfrak{A}$ and $\mathfrak{A}'$ be quaternion algebras over K with corresponding norm forms N and N'. Then the following statements are equivalent:*

1. $\mathfrak{A} \cong \mathfrak{A}'$
2. *N and N' are isometric*
3. *N_0 and N_0' are isometric*

Moreover, $\mathfrak{A}$ a division algebra is equivalent to N being anisotropic, which in turn is equivalent to N_0 being anisotropic.

This correspondence between quaternion algebras and quaternary and ternary quadratic forms will be used in the sequel. Especially a refinement of the correspondence with ternary quadratic forms will be presented in detail in Section 4 and used in the following sections.

We now specialise to the fields of our main interest. So from now on assume that K is an algebraic number field. If ν is a place on K, then $\mathfrak{A}_\nu = \mathfrak{A} \otimes_K K_\nu$ is a quaternion algebra over K_ν.

Every quaternary quadratic form Q over K represents 1 iff it is not negative definite at any real place [13, Satz 19]. Hence, we may conclude that

$$Q \cong \langle 1 \rangle \perp \langle a \rangle \perp \langle b \rangle \perp \langle c \rangle ,$$

for some $a, b, c \in K$, if Q is not negative definite at any real place. If the discriminant of Q is equal to 1, then we may assume that $c = ab$ and we conclude from (2.4) the following:

(2.5) Proposition. *Let K be an algebraic number field. Then there is a one-to-one correspondence between*

1. *isomorphism classes of quaternion algebras over K,*
2. *isometry classes of quaternary quadratic forms over K with discriminant equal to 1, which are not negative definite at any real place.*

Furthermore, if ν is a place on K, then there is a one-to-one correspondence between

1. *isomorphism classes of quaternion algebras over K_ν,*
2. *isometry classes of quaternary quadratic forms over K_ν with discriminant equal to 1, which are not negative definite if $K_\nu = \mathbb{R}$.*

Hence, one can use the classification of quadratic forms in order to classify quaternion algebras up to isomorphism.

The classification of quadratic forms up to isometry over $\mathbb{Q}$ was first made by Minkowski in [21]. This classification was later simplified and generalised to arbitrary algebraic number fields by Hasse in [12] and [14]. Using the results of Hasse and (2.5) one derives the following two theorems on the classification of quaternion algebras.

(2.6) Theorem. *Let K_ν be a completion of a number field K. Then there are exactly two quaternion algebras over K_ν up to isomorphism, $M_2(K_\nu)$ and a division algebra.*

The division algebra over K_ν will be denoted by $\mathbb{H}_\nu$. We say that $\mathfrak{A}$ is ramified at ν if $\mathfrak{A}_\nu \cong \mathbb{H}_\nu$, otherwise $\mathfrak{A}$ is said to split at ν. If ν is a real place, then one often uses definite/indefinite instead of ramified/split, especially if $K = \mathbb{Q}$.

(2.7) Theorem. *Let K be an algebraic number field and $\mathfrak{A}$ and $\mathfrak{A}'$ two quaternion algebras over K. Then the following statements are equivalent:*

(1) $\mathfrak{A} \cong \mathfrak{A}'$,
(2) $\mathfrak{A}_\nu \cong \mathfrak{A}'_\nu$, $\forall \nu \in \Omega$,
(3) $\mathfrak{A}$ *and* $\mathfrak{A}'$ *are ramified at the same places.*

Moreover, $\mathfrak{A}$ is always ramified at an even number of places. Conversely, given an even number of places, it is always possible to find a quaternion algebra which is ramified at exactly these places.

The (reduced) discriminant, $d(\mathfrak{A})$, of a quaternion algebra $\mathfrak{A}$ is defined to be the product of the prime ideals $\mathfrak{p}$ at which $\mathfrak{A}$ is ramified. This is a well-defined invariant of the isomorphism classes by (2.7). If $K = \mathbb{Q}$, then the discriminant determines the isomorphism class, but for other fields one clearly also needs information on the infinite ramifications.

With this classification at hand, it is of course of interest to be able to easily determine the ramifications of a given quaternion algebra over K. It turns out that the Hilbert symbol solves the problem. If $\nu \in \Omega$ and $a, b \in K_\nu^*$, then the Hilbert symbol $(a,b)_\nu$ in K_ν is defined by

$$(a,b)_\nu = \begin{cases} 1, & \text{if } X^2 - aY^2 - bZ^2 \text{ is isotropic over } K_\nu \\ -1, & \text{otherwise.} \end{cases}$$

We remark that if N_0 is the restricted norm form on $(a,b)_K$, then

$$abN_0 \cong \langle -a \rangle \perp \langle -b \rangle \perp \langle 1 \rangle .$$

From this we conclude that N_0 is isotropic at ν iff $(a,b)_\nu = 1$. Hence, it is immediate from (2.4) that $(a,b)_K$ is ramified at ν iff $(a,b)_\nu = -1$. For a proof of the following properties of the Hilbert symbol, see [23, §63,71]. We remark, that property (d) is exactly what is needed to prove that $\mathfrak{A}$ is always ramified at an even number of places.

(2.8) Proposition. *The Hilbert symbol satisfies the following:*

(a) $(a,bc)_\nu = (a,b)_\nu (a,c)_\nu$, $(a,-a)_\nu = 1$ *and* $(a,b^2)_\nu = 1$.
(b) *If ν is real, then $(a,b)_\nu = -1$ iff $a < 0$ and $b < 0$.*
(c) *If $\mathfrak{p}$ is a non-dyadic prime, then* $\begin{cases} (a,b)_\mathfrak{p} = 1, & \textit{if } a,b \in R_\mathfrak{p}^* \\ (a,\mathfrak{p})_\mathfrak{p} = (\frac{a}{\mathfrak{p}}), & \textit{if } a \in R_\mathfrak{p}^*, \end{cases}$
where $(\frac{a}{\mathfrak{p}}) = 1$ if a is a square modulo $\mathfrak{p}$ and -1 otherwise.
(d) $\prod_{\nu \in \Omega} (a,b)_\nu = 1$, $\forall a, b \in K^*$.

The dyadic case is more difficult in general. If there is only one dyadic prime, then (2.8)(d) solves the problem, but otherwise we might have to do some calculations. However, it is possible to prove the following general result. If π is a prime element in $R_\mathfrak{p}$, $\epsilon \in R_\mathfrak{p}^*$ and $\alpha \in R_\mathfrak{p}^*$ is an element of quadratic defect $\delta_\mathfrak{p}(\alpha) = 4R_\mathfrak{p}$, then [23, 63:11a]

$$(\pi, \alpha)_\mathfrak{p} = -1 \text{ and } (\epsilon, \alpha)_\mathfrak{p} = 1. \tag{2.9}$$

(An element is of quadratic defect $4R_\mathfrak{p}$, if it is not a square but congruent to a square modulo $4R_\mathfrak{p}$.)

The concluding result of this section gives an answer to the question: Given the ramifications of a quaternion algebra $\mathfrak{A}$ over K, how to find $a, b \in R$ such that $\mathfrak{A} \cong (a, b)_K$? If $d(\mathfrak{A}) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$, then let n_i be positive integers such that $\mathfrak{I} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ is principal. A generator of the principal ideal $\mathfrak{I}$ will be called a representative of $d(\mathfrak{A})$. The proposition below will give an answer to our question when $d(\mathfrak{A}) = R$ or $d(\mathfrak{A})$ has a representative $\mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$, such that all integers n_i are odd, in particular, when $d(\mathfrak{A})$ is principal. It is also a very explicit proof of the last conclusion in (2.7) in this restricted case. In principal, it follows the same idea as in Hasse's original proof.

(2.10) Proposition. *Let $\mathfrak{A}$ be a quaternion algebra over an algebraic number field K. Suppose that $d(\mathfrak{A}) = R$ or $d(\mathfrak{A}) = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ has a representative $(d) = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$, such that all integers n_i are odd. Then choose $a \in R$ to be a generator of a prime ideal satisfying* $\gcd(a, d) = 1$ *and*

$$\begin{cases} a < 0, & \text{at } \nu \text{ real, if } \mathfrak{A} \text{ ramified at } \nu, \\ a > 0, & \text{at } \nu \text{ real, if } \mathfrak{A} \text{ split at } \nu, \\ (\frac{a}{\mathfrak{p}}) = -1, & \text{at } \mathfrak{p} \text{ non-dyadic, if } \mathfrak{A} \text{ ramified at } \mathfrak{p}, \\ \delta_\mathfrak{p}(a) = 4R_\mathfrak{p}, & \text{at all dyadic primes } \mathfrak{p}. \end{cases} \tag{2.11}$$

Then $\mathfrak{A} \cong (a, -d)_K$.

Proof. The existence of such an a is assured by the generalisation to arbitrary number fields of Dirichlet's Theorem on primes in linear progressions [22, Cor.4, p.360]. The ramifications of $\mathfrak{A}$ and $(a, -d)_K$ obviously agree at all real places. If $\mathfrak{p}$ is a non-dyadic prime dividing d, then (2.8) implies that

$$(a, -d)_\mathfrak{p} = (a, \mathfrak{p})_\mathfrak{p} = (\frac{a}{\mathfrak{p}}) = -1.$$

This is where it is necessary to have $\mathfrak{p}$ to an odd power in (d). The only other non-dyadic prime at which $(a, -d)_K$ could possibly be ramified is a, since $a, d \in R_\mathfrak{p}^*$ for all other non-dyadic primes.

If $\mathfrak{p}$ is a dyadic prime, then (2.9) implies that $(a, -d)_\mathfrak{p} = -1$ iff $\mathfrak{p}|d$. Hence, the ramifications of $\mathfrak{A}$ and $(a, -d)_K$ differ at most at a and then (2.7) implies that $\mathfrak{A} \cong (a, -d)_K$. $\square$

For the field of rational numbers (2.10) simplifies to (compare [19])

(2.12) Corollary. *Let $\mathfrak{A}$ be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $d = p_1 \cdots p_{2r}$. Choose p to be a prime such that $p \equiv 5 \ (mod\ 8)$ and $(\frac{p}{p_i}) = -1,\ \forall p_i > 2$. Then $\mathfrak{A} \cong (p, d)_{\mathbb{Q}} \cong (p, -d)_{\mathbb{Q}}$.*

(2.13) Corollary. *Let $\mathfrak{A}$ be a definite quaternion algebra over $\mathbb{Q}$ with discriminant $d = p_1 \cdots p_{2r-1}$. Choose p to be a prime such that $p \equiv 3 \ (mod\ 8)$ and $(\frac{p}{p_i}) = -1,\ \forall p_i > 2$. Then $\mathfrak{A} \cong (-p, -d)_{\mathbb{Q}}$.*

3. Quaternion orders

In this section, R will be a Dedekind ring with field of fractions K.

An R-lattice Λ on $\mathfrak{A}$ is a finitely generated R-module such that $K\Lambda = \mathfrak{A}$. We remark that it is not always possible to find an R-basis of Λ. However, it is always possible to find a basis $e_1, \ldots, e_4$ of $\mathfrak{A}$ and an R-ideal $\mathfrak{a}$, such that $\Lambda = \mathfrak{a}e_1 \oplus Re_2 \oplus Re_3 \oplus Re_4$ [23, 81:5]. Let Λ and Λ' be two lattices on $\mathfrak{A}$. The index, $[\Lambda : \Lambda']$, of Λ' in Λ is defined to be the R-ideal generated by $\det(\varphi)$ of all linear transformations $\varphi : \mathfrak{A} \longrightarrow \mathfrak{A}$, such that $\varphi(\Lambda) \subseteq \Lambda'$. In particular, if Λ and Λ' both have R-bases, then $[\Lambda : \Lambda']$ is the determinant of the matrix which takes a basis of Λ into a basis of Λ'. Given an R-lattice Λ on $\mathfrak{A}$, we define its dual $\Lambda^{\#}$ to be

$$\Lambda^{\#} = \{q \in \mathfrak{A} : Tr(q\Lambda) \subseteq R\}.$$

This is again an R-lattice on $\mathfrak{A}$.

(3.1) Definition. An (R-)order $\mathcal{O}$ in a quaternion algebra $\mathfrak{A}$ is an R-lattice on $\mathfrak{A}$, which is also a ring containing R.

For the rest of this paper $\mathcal{O}$ will always denote a quaternion order. We remark that if $x \in \mathcal{O}$, then x is integral over R, so $Tr(x), N(x) \in R$.

If Λ is a lattice on $\mathfrak{A}$, then we define the left (right) order $\mathcal{O}_l(\Lambda)$ ($\mathcal{O}_r(\Lambda)$) of Λ to be

$$\mathcal{O}_l(\Lambda) = \{x \in \mathfrak{A} : x\Lambda \subseteq \Lambda\} \text{ and } \mathcal{O}_r(\Lambda) = \{x \in \mathfrak{A} : \Lambda x \subseteq \Lambda\}.$$

It is easy to verify that both $\mathcal{O}_l(\Lambda)$ and $\mathcal{O}_r(\Lambda)$ are orders in $\mathfrak{A}$.

A left (right) ideal of an order $\mathcal{O}$ is a lattice Λ on $\mathfrak{A}$ such that $\mathcal{O}\Lambda \subseteq \Lambda$ ($\Lambda\mathcal{O} \subseteq \Lambda$). The ideal is called a two-sided ideal of $\mathcal{O}$, if it is both a left and right ideal of $\mathcal{O}$. If Λ is any lattice, then obviously Λ is a left $\mathcal{O}_l(\Lambda)$-ideal and a right $\mathcal{O}_r(\Lambda)$-ideal. An ideal Λ is a principal $\mathcal{O}$-ideal, if there is $x \in \mathfrak{A}$ such that $\Lambda = \mathcal{O}x$.

The most important invariant of a quaternion order is the (reduced) discriminant. It is defined as follows. Let $\mathfrak{I}$ be the ideal generated by all $\det(Tr(x_i\bar{x}_j))$, where $x_1, \ldots, x_4 \in \mathcal{O}$. It is easy to prove that $\mathfrak{I}$ is the square of an ideal. The (reduced) discriminant, $d(\mathcal{O})$, of $\mathcal{O}$ is defined to be the square root of $\mathfrak{I}$. If $\mathcal{O}$ and $\mathcal{O}'$ are two orders, then the discriminants satisfy

$$d(\mathcal{O}) = d(\mathcal{O}') \cdot [\mathcal{O}' : \mathcal{O}]. \tag{3.2}$$

(3.3) Example. Let $\mathfrak{A} = (a,b)_K$ be a quaternion algebra. Suppose that $a, b \in R$. This is no restriction, since $(a,b)_K \cong (ax^2, by^2)_K$ for all $x, y \in K$. Then

$$\mathcal{O} = R + Ri + Rj + Rij,$$

where i, j satisfy (2.2) is an order in $\mathfrak{A}$. We will denote this order by $(a,b)_R$. The discriminant of $(a,b)_R$ is

$$(\det(Tr(x_i \bar{x}_j)))^{\frac{1}{2}},$$

where $\{x_1, \ldots, x_4\} = \{1,\, i,\, j,\, ij\}$. A simple calculation shows that the matrix is diagonal and $d(\mathcal{O}) = (4ab)$.

□

A maximal order is an order, which is not strictly contained in any other order. The concept of maximal orders is more complicated than in the commutative case, since there is more than one maximal order in general. This complication occurs since $\{x \in \mathfrak{A} : N(x), Tr(x) \in R\}$ is not always a ring. However, the discriminants of all maximal orders in a quaternion algebra always agree (see (5.1) and (6.3)).

The task to classify quaternion orders is of course much more complicated than for quaternion algebras. In particular, there is no analogue of (2.7), that is, isomorphism at all places no longer implies global isomorphism. However, the investigation of $\mathcal{O}_\nu = \mathcal{O} \otimes_R R_\nu$ is still essential and useful in order to classify quaternion orders $\mathcal{O}$ in algebras over algebraic number fields (see Section 6).

In order to get a structure on the set of orders in a quaternion algebra, we introduce special classes of orders. An order $\mathcal{O}$ is called a Gorenstein order if $\mathcal{O}^\#$ is projective as $\mathcal{O}$-module, and it is called a Bass order if every order in $\mathfrak{A}$ containing $\mathcal{O}$ is a Gorenstein order. For an arbitrary order $\mathcal{O}$, there is a unique Gorenstein order $G(\mathcal{O})$ (called the Gorenstein closure) and a unique R-ideal $b(\mathcal{O}) \subseteq R$ (the Brandt invariant) such that [6, (1.4)]

$$\mathcal{O} = R + b(\mathcal{O})G(\mathcal{O}). \tag{3.4}$$

In [9], Eichler introduced so called primitive orders. An order is called primitive, if it contains a maximal order of a quadratic subfield of $\mathfrak{A}$. It is easy to show that every primitive order is a Bass order. However, whether every Bass order is primitive is an open question in general. It is true in the local case by [8, (1.11)] and in the case of rational orders by [9, Satz 8].

An order $\mathcal{O}$ is called a hereditary order if every $\mathcal{O}$-ideal is $\mathcal{O}$-projective. It is well-known that hereditary orders are exactly those with square free discriminant [27, (39.14)]. These classes of orders obviously satisfy the inclusions

$$\{\text{Gorenstein}\} \supset \{\text{Bass}\} \supset \{\text{Hereditary}\} \supseteq \{\text{Maximal}\}.$$

4. Ternary quadratic forms and quaternion orders

Before we continue the investigation of quaternion orders, we will in this section refine the correspondence between quaternions and ternary quadratic forms. We will have to assume that R is a principal ideal domain, since otherwise we will have some complications.

If f is a non-degenerate ternary quadratic form integral over R, then define $C_0(f)$ to be the even Clifford algebra over R associated to f. Then $C_0(f)$ is an order in a quaternion algebra over K. If

$$f = \sum_{1 \le i \le j \le 3} a_{ij} X_i X_j,$$

then a direct computation shows that $C_0(f)$ has an R-basis $1, e_1, e_2, e_3$ such that:

$$\begin{aligned} e_i^2 &= a_{jk} e_i - a_{jj} a_{kk}, \\ e_i e_j &= a_{kk}(a_{ij} - e_k), \\ e_j e_i &= a_{1k} e_1 + a_{2k} e_2 + a_{3k} e_3 - a_{ik} a_{jk}, \end{aligned} \tag{4.1}$$

where (i,j,k) is an even permutation of $(1,2,3)$. From this we get that the norm form for $C_0(f)$ in this basis is

$$Q = X_0^2 + \sum_{(i,j,k)} \left[a_{ij} X_0 X_k + a_{ii} a_{jj} X_k^2 + (a_{ik} a_{jk} - a_{ij} a_{kk}) X_i X_j \right], \tag{4.2}$$

where the sum is over all even permutations (i,j,k) of $(1,2,3)$. It is trivial to check from the relations (4.1) that if $\epsilon \in R^*$, then $C_0(f) = C_0(\epsilon f)$.

Conversely, assume that $\mathcal{O} = \langle 1, e_1, e_2, e_3 \rangle$ is an R-order in a quaternion algebra $\mathfrak{A}$. Then $\Lambda = \mathcal{O}^\# \cap \mathfrak{A}_0$ is a 3-dimensional R-lattice on $\mathfrak{A}_0$. If $\mathcal{O}^\# = \langle f_0, f_1, f_2, f_3 \rangle$ where $\{f_i\}$ is the dual basis of $\{e_i\}$, then $\Lambda = \langle f_1, f_2, f_3 \rangle$. Given this, we define a ternary quadratic form $f_{\mathcal{O}}$ associated to $\mathcal{O}$ by

$$f_{\mathcal{O}} = d(\mathcal{O}) \cdot N(X_1 f_1 + X_2 f_2 + X_3 f_3).$$

Here multiplication by the (principal) ideal $d(\mathcal{O})$ is understood as multiplication by a generator of $d(\mathcal{O})$. Hence, $f_{\mathcal{O}}$ is only defined up to multiplication by units in R.

This construction is due to Brzezinski, and it is a generalisation of a construction originally made by Brandt in [2]. This in turn is a generalisation of a result in [20]. The construction of Brandt was investigated and clarified by Peters in [24]. Peters proved that it gives exactly all Gorenstein orders under the restriction $\mathrm{char} K \neq 2$. This restriction was eliminated in [5]. Notice that the modified correspondence given here gives all orders. Since there is no complete proof of the following theorem in the literature, we will give one here which is almost self-contained. We will closely follow the ideas in [5].

(4.3) Theorem. *Let R be a principal ideal domain. The maps $f \mapsto C_0(f)$ and $\mathcal{O} \mapsto f_{\mathcal{O}}$ are inverses to each other and the discriminants satisfy*

$d(\mathcal{O}) = (d(f_{\mathcal{O}}))$. Furthermore, the maps give a bijection between similarity classes of non-degenerate ternary quadratic forms integral over R and isomorphism classes of quaternion R-orders.

Proof. First we prove that $C_0(f_{\mathcal{O}}) = \mathcal{O}$. Suppose that $\mathcal{O} = \langle x_0, x_1, x_2, x_3\rangle$ and $\mathcal{O}^{\#} = \langle f_0, f_1, f_2, f_3\rangle$, where $\{f_i\}$ is the dual basis of $\{x_i\}$ and $x_0 = 1$. By definition, these bases satisfy $Tr(x_i\bar{f}_j) = \delta_{ij}$. In particular, $Tr(f_0) = 1$ and $Tr(f_i) = 0$, if $i > 0$. It is straightforward to check that

$$x_i = Tr(f_1f_2f_3)^{-1}\left(Tr(f_jf_k\bar{f}_0) - f_jf_k\right), \tag{4.4}$$

where (i, j, k) is an even permutation of $(1, 2, 3)$. For example,

$$\begin{aligned} Tr\left((Tr(f_jf_k\bar{f}_0) - f_jf_k)\bar{f}_i\right) &= Tr(f_jf_k\bar{f}_0)Tr(\bar{f}_i) - Tr(f_jf_k\bar{f}_i) \\ &= Tr(f_jf_kf_i) = Tr(f_if_jf_k) = Tr(f_1f_2f_3), \end{aligned}$$

since $(y_1, y_2, y_3) \longmapsto Tr(y_1y_2y_3)$ is trilinear and alternating on $\mathfrak{A}_0$, and $\bar{y} = -y$ on $\mathfrak{A}_0$. By the proof of [5, (3.2)], $d = Tr(f_1f_2f_3)^{-1}$ is a generator of $d(\mathcal{O})$ and df_jf_k is integral over R. Hence, we can rewrite (4.4) as

$$df_jf_k = d\cdot Tr(f_jf_k\bar{f}_0) - x_i \tag{4.5}$$

and conclude that $d\cdot Tr(f_jf_k\bar{f}_0) \in R$.

From the definition of $f_{\mathcal{O}}$, we get that

$$f_{\mathcal{O}} = d\cdot\sum_{i\le j} a_{ij}X_iX_j, \text{ where } a_{ij} = \begin{cases} Tr(f_i\bar{f}_j), & \text{if } i \ne j \\ N(f_i), & \text{if } i = j. \end{cases} \tag{4.6}$$

Now it is straightforward to check that $\{e_i = df_j\bar{f}_k\}$, with (i, j, k) an even permutation of $(1, 2, 3)$, satisfy the relations (4.1) with $\{a_{ij}\}$ as in (4.6). Hence, we can identify $C_0(f_{\mathcal{O}})$ with the order $\mathcal{O}' = \langle 1, df_1\bar{f}_2, df_2\bar{f}_3, df_3\bar{f}_1\rangle$. But since $df_j\bar{f}_k = -df_jf_k = x_i - d\cdot Tr(f_jf_k\bar{f}_0)$, we get $\mathcal{O} = \mathcal{O}' = C_0(f_{\mathcal{O}})$.

To prove the other direction, that is, $f_{C_0(f)} = f$ is a straightforward calculation. All one has to do is to determine the dual basis f_0, f_1, f_2, f_3 of the basis satisfying (4.1), and then calculate $N(X_1f_1 + X_2f_2 + X_3f_3)$. Of course, a computer with basic knowledge of non-commutative algebra is of great help.

The fact that isometric forms give isomorphic Clifford algebras follows directly from the universal property of Clifford algebras. Hence $f \sim g$ implies $C_0(f) \cong C_0(g)$, since as we remarked before, $C_0(f) = C_0(\epsilon f)$ if $\epsilon \in R^*$.

Conversely, let $\mathcal{O}_1$ and $\mathcal{O}_2$ be isomorphic orders in $\mathfrak{A}$. Then there is an $a \in \mathfrak{A}$ such that $\mathcal{O}_1 = a^{-1}\mathcal{O}_2a$ and $\Lambda_1 = a^{-1}\Lambda_2a$, where $\Lambda_i = \mathcal{O}_i^{\#} \cap \mathfrak{A}_0$. But $x \longmapsto a^{-1}xa$ is an isometry of $\mathfrak{A}_0$ with respect to N_0. Hence $f_{\mathcal{O}_1}$ is isometric to $f_{\mathcal{O}_2}$.

Finally, it is an easy calculation to show that $d(\mathcal{O}) = (d(f_{\mathcal{O}}))$. $\square$

(4.7) Proposition. *Let f be a non-degenerate ternary quadratic form integral over R and $\mathcal{O} = C_0(f)$. If $f = b\cdot g$, where $b \in R$ and g is primitive,*

then the Brandt invariant of $\mathcal{O}$ is equal to (b) and the Gorenstein closure of $\mathcal{O}$ is equal to $C_0(g)$. In particular, $\mathcal{O}$ is a Gorenstein order iff f is primitive.

Proof. It follows immediately from the relations (4.1) that if $f = b \cdot g$, then $C_0(f) = R + b \cdot C_0(g)$. The proposition follows from this and (3.4). □

5. Quaternion orders in the $\mathfrak{p}$-adic case

In this section K will be a $\mathfrak{p}$-adic field with ring of integers R with prime $\mathfrak{p} = (\pi)$. We will use the correspondence in Section 4 and classifications of ternary quadratic forms to give a very explicit classification of quaternion orders. It is much more elaborate in the dyadic than in the non-dyadic case. Therefore we will restrict to 2-adic fields in the dyadic case, that is, fields in which (2) is prime.

The discussion in this section will show that the maximal orders are unique up to isomorphism in the $\mathfrak{p}$-adic case. However, in the case of a division algebra one can say even more. Therefore we include the following well-known result [28, ch. II]:

(5.1) Proposition. *Let $\mathcal{O}$ be a maximal order in a quaternion algebra $\mathfrak{A}$ over a local field K, and let $\mathfrak{p}$ be the maximal ideal in R. If $\mathfrak{A} \cong M(2, K)$, then $\mathcal{O}$ is conjugate to $M(2, R)$, so $d(\mathcal{O}) = R$, and if $\mathfrak{A}$ is a division algebra, then $\mathcal{O}$ is unique and $d(\mathcal{O}) = \mathfrak{p}$.*

We recall an invariant of orders in the local case introduced by Eichler in [9]. Let k be the residue class field of K and $J(\mathcal{O})$ the Jacobson radical of $\mathcal{O}$. If $\mathcal{O} \not\cong M_2(R)$, then the Eichler invariant $e(\mathcal{O})$ is defined to be:

$$(5.2)\qquad e(\mathcal{O}) = \begin{cases} 1, & \text{if } \mathcal{O}/J(\mathcal{O}) \cong k \oplus k \\ 0, & \text{if } \mathcal{O}/J(\mathcal{O}) \cong k \\ -1, & \text{if } \mathcal{O}/J(\mathcal{O}) \text{ is a quadratic field extension of } k. \end{cases}$$

Eichler also showed how to compute $e(\mathcal{O})$ easily. If $x \in \mathfrak{A}$, then the discriminant of x is defined to be $\Delta(x) = Tr(x)^2 - 4N(x)$. The Eichler invariant can be determined using the following result [9, Satz 10]:

(5.3) Lemma. *The Eichler invariant $e(\mathcal{O})$ satisfies:*

1. *If $e(\mathcal{O}) = 0$, then $(\frac{\Delta(x)}{\mathfrak{p}}) = 0\ \forall x \in \mathcal{O}$.*
2. *If $e(\mathcal{O}) = 1$, then $(\frac{\Delta(x)}{\mathfrak{p}}) \neq -1\ \forall x \in \mathcal{O}$, and $\exists x \in \mathcal{O} : (\frac{\Delta(x)}{\mathfrak{p}}) = 1$.*
3. *If $e(\mathcal{O}) = -1$, then $(\frac{\Delta(x)}{\mathfrak{p}}) \neq 1\ \forall x \in \mathcal{O}$, and $\exists x \in \mathcal{O} : (\frac{\Delta(x)}{\mathfrak{p}}) = -1$.*

Here $(\frac{a}{\mathfrak{p}}) = 0$ iff $a \in \mathfrak{p}$.

We remark that it is easy to show using (3.4) and (5.3) that if $\mathcal{O}$ is not a Bass order, then $e(\mathcal{O}) = 0$.

Let f be a ternary quadratic form integral over R. We can restrict to f primitive, since if $f_1 = b_1 \cdot g_1$ and $f_2 = b_2 \cdot g_2$ with $b_i \in R$ and g_i primitive forms, then $f_1 \sim f_2$ iff $g_1 \sim g_2$ and the ideals (b_1) and (b_2) are equal. There is a classification of quadratic lattices in the local case in [23, §§92,93], which we will make use of below.

In the non-dyadic case every form is isometric to a diagonal form, but this is not true in the 2-adic case. We define the 2-dimensional quadratic forms H and J with the corresponding matrices

$$H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } J = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since we may multiply f by elements in R^*, we can conclude the following from the results in [23]:

(5.4) Proposition. *Let f be a primitive ternary quadratic form over R. If $\mathfrak{p}$ is a non-dyadic prime, then*

$$f \sim \langle 1 \rangle \perp \langle \delta\pi^r \rangle \perp \langle \epsilon\pi^s \rangle \,.$$

If $\mathfrak{p} = (2)$, then $f \sim f_i$ for some $1 \leq i \leq 5$, where

$$\begin{aligned} f_1 &= \langle 1 \rangle \perp \langle \delta 2^r \rangle \perp \langle \epsilon 2^s \rangle \,, \\ f_2(r) &= \langle 1 \rangle \perp 2^r H, \\ f_3(r) &= H \perp \langle 2^r \rangle \,, \\ f_4(r) &= \langle 1 \rangle \perp 2^r J, \\ f_5(r) &= J \perp \langle 2^r \rangle \,. \end{aligned}$$

Here $\epsilon, \delta \in R^*$ and $0 \leq r \leq s$.

The quadratic forms in (5.4) will be called standard forms.

(5.5) Remark. The only cases when any of the non-diagonal forms $f_2, \ldots, f_5$ are similar to a diagonal form are

$$\langle 1 \rangle \perp 2H \sim \langle 1 \rangle \perp \langle 1 \rangle \perp \langle -1 \rangle \text{ and } \langle 1 \rangle \perp 2J \sim \langle 1 \rangle \perp \langle 1 \rangle \perp \langle 1 \rangle \,. \tag{5.6}$$

Furthermore, we also have

$$f_2(0) = f_3(0) \sim f_4(0) = f_5(0) \text{ and } f_2(2) \sim f_4(2). \tag{5.7}$$

However, $f_i \not\sim f_j$ if $i \neq j$ in all cases except (5.6) and (5.7). □

The following proposition gives simple criteria on f whether $C_0(f)$ is a Bass order or not.

(5.8) Proposition. *If $\mathfrak{p}$ is non-dyadic and $f \sim \langle 1 \rangle \perp \langle \delta\pi^r \rangle \perp \langle \epsilon\pi^s \rangle$ with $r \leq s$, then $C_0(f)$ is a Bass order iff $r \leq 1$.*

If $\mathfrak{p} = (2)$, then $C_0(f)$ is a Bass order iff f is similar to any of the forms

$$\begin{aligned} &\langle 1 \rangle \perp \langle \delta \rangle \perp \langle \epsilon 2^r \rangle \,, \ \textit{with } \delta \equiv 1 \ (\textit{mod } 4) \textit{ or } r \leq 1, \\ &\langle 1 \rangle \perp \langle \delta 2 \rangle \perp \langle \epsilon 2^r \rangle \,, \\ &f_3(r) \textit{ or } f_5(r). \end{aligned}$$

Proof. It follows from (4.2) that if $f = \langle 1 \rangle \perp \langle \delta\pi^r \rangle \perp \langle \epsilon\pi^s \rangle$, then the norm form in $\mathcal{O} = C_0(f)$ is given by

$$N = \langle 1 \rangle \perp \langle \delta\pi^r \rangle \perp \langle \epsilon\pi^s \rangle \perp \left\langle \delta\epsilon\pi^{r+s} \right\rangle .$$

It is easy to check from this that $\mathcal{O}$ contains a primitive element (and hence is a Bass order) exactly in the cases in the proposition. The non-diagonal cases are analogous. □

If we have a standard form f, then it is easy to calculate the Eichler invariant of $C_0(f)$. A direct computation using (5.3) gives:

(5.9) Proposition. *If $\mathcal{O} = C_0(f)$, then the Eichler invariant $e(\mathcal{O})$ satisfies:*

If $\mathfrak{p}$ is non-dyadic and $f \sim \langle 1\rangle \perp \langle \delta\pi^r\rangle \perp \langle \epsilon\pi^s\rangle$ with $r \le s$, then:

$$e(\mathcal{O}) = 1 \quad \textit{iff} \quad r = 0,\ s \ge 1 \ \textit{and}\ (\frac{-\delta}{\mathfrak{p}}) = 1.$$
$$e(\mathcal{O}) = -1 \quad \textit{iff} \quad r = 0,\ s \ge 1 \ \textit{and}\ (\frac{-\delta}{\mathfrak{p}}) = -1.$$

If $\mathfrak{p} = (2)$, then:

$$e(\mathcal{O}) = 1 \quad \textit{iff} \quad f \sim H \perp \langle 2^r\rangle\,,\ r \ge 1.$$
$$e(\mathcal{O}) = -1 \quad \textit{iff} \quad f \sim J \perp \langle 2^r\rangle\,,\ r \ge 1.$$

There are descriptions of all orders in quaternion algebras over local fields in both [6] and [17]. In [6], which is more ring-theoretic in nature, there are also detailed information on relations between orders. Now we will, as a complement to these descriptions, give a set of primitive ternary quadratic forms M_1 such that $M' = \{C_0(f) : f \in M_1\}$ is a set of representatives of all isomorphism classes of Gorenstein orders. Then

$$M = \{C_0(\pi^b f) : f \in M_1,\ b \ge 0\}$$

is a set of representatives of all isomorphism classes of orders. One of the advantages of our description is that it is very well suited for explicit calculations, since f in standard form gives a convenient basis of $C_0(f)$.

We start with the non-dyadic case. If

$$f_1 = \langle 1\rangle \perp \langle \delta_1\pi^{r_1}\rangle \perp \langle \epsilon_1\pi^{s_1}\rangle \sim f_2 = \langle 1\rangle \perp \langle \delta_2\pi^{r_2}\rangle \perp \langle \epsilon_2\pi^{s_2}\rangle\,,$$

then $r_1 = r_2$ and $s_1 = s_2$. Conversely, if $r_1 = r_2$, $s_1 = s_2$, $(\frac{\delta_1}{\mathfrak{p}}) = (\frac{\delta_2}{\mathfrak{p}})$ and $(\frac{\epsilon_1}{\mathfrak{p}}) = (\frac{\epsilon_2}{\mathfrak{p}})$, then $f_1 \sim f_2$. Hence, we have at most 4 different classes given r and s. But if $r = 0$ or $r = s$, then the number of classes is divided by 2 [23, 92:1]. This is summarised in:

(5.10) Proposition. *Let $\mathfrak{p}$ be a non-dyadic prime and let $\mathcal{O}$ be a Gorenstein order in a quaternion algebra over K. Then $\mathcal{O} \cong C_0(f)$, where f is uniquely chosen among the following quadratic forms:*

1. $\langle 1\rangle \perp \langle 1\rangle \perp \langle 1\rangle$
2. $\langle 1\rangle \perp \langle -\epsilon_1\rangle \perp \langle \pi^s\rangle\,,\ s \ge 1$
3. $\langle 1\rangle \perp \langle \pi^r\rangle \perp \langle \epsilon_1\pi^r\rangle\,,\ r \ge 1$
4. $\langle 1\rangle \perp \langle \epsilon_1\pi^r\rangle \perp \langle \epsilon_2\pi^s\rangle\,,\ s > r \ge 1$

Here ϵ_1 and ϵ_2 are to be chosen arbitrarily in $\{1, \delta\}$, where $(\frac{\delta}{\mathfrak{p}}) = -1$.

With (5.8) and (5.10) at hand, we are able to determine the number of isomorphism classes of Bass and Gorenstein orders with given discriminant. Furthermore, if $t(n)$ is the number of isomorphism classes of orders with

discriminant $\mathfrak{p}^n$ and $g(n)$ is the number of isomorphism classes of Gorenstein orders with discriminant $\mathfrak{p}^n$, then

$$t(n) = g(n) + t(n-3). \tag{5.11}$$

The reason for this is that an order $\mathcal{O}$ is either Gorenstein or else $\mathcal{O} = R+\mathfrak{p}\mathcal{O}'$ with $[\mathcal{O}' : \mathcal{O}] = \mathfrak{p}^3$. Moreover, if $\mathcal{O}_1 = R + \mathfrak{p}\mathcal{O}_1'$ and $\mathcal{O}_2 = R + \mathfrak{p}\mathcal{O}_2'$, then $\mathcal{O}_1 \cong \mathcal{O}_2$ iff $\mathcal{O}_1' \cong \mathcal{O}_2'$. The linear recursion (5.11) is easy to solve, since $g(n)$ proves to be more or less a linear function. We summarise everything in Table 1, in which we separate orders in $\mathbb{H}_\mathfrak{p}$ and $M_2 = M_2(K)$. The two variables n and α in Table 1 satisfy

$$n \geq 3 \text{ and } \alpha = \begin{cases} \frac{1}{3}, & \text{if } n \equiv 0 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

	BASS		GOR.		TOTALLY	
Disc.	$\mathbb{H}_\mathfrak{p}$	M_2	$\mathbb{H}_\mathfrak{p}$	M_2	$\mathbb{H}_\mathfrak{p}$	M_2
1	0	1	0	1	0	1
$\mathfrak{p}$	1	1	1	1	1	1
$\mathfrak{p}^2$	1	3	1	3	1	3
$\mathfrak{p}^n, n$ odd	3	3	n	n	$\frac{1}{8}(n^2+4n+3)$	$\frac{1}{24}(5n^2+12n+7)+\alpha$
$\mathfrak{p}^n, n$ even	2	4	$\frac{n}{2}$	$\frac{3n}{2}$	$\frac{1}{8}(n^2+2n)$	$\frac{1}{24}(5n^2+18n+16)+\alpha$

TABLE 1. The number of isomorphism classes of Bass, Gorenstein and arbitrary orders in $\mathbb{H}_\mathfrak{p}$ and $M_2 = M_2(K)$, when $\mathfrak{p}$ is non-dyadic.

If we restrict to Bass orders, then we can also easily draw conclusions on the relations between the classes of orders by using results first proved by Eichler in [9, Satz 12]. These relations on inclusions between orders were generalised in [6]. We illustrate this by the trees in Figure 1. Every node in a tree represents an isomorphism class of Bass orders. Different isomorphism classes with the same discriminant are on the same level. There is an edge between nodes n_1 and n_2 iff given $\mathcal{O}_1 \in n_1$, then there exists $\mathcal{O}_2 \in n_2$ such that $\mathcal{O}_1$ is a maximal suborder in $\mathcal{O}_2$ or vice versa. The numbers at the bottom are the Eichler invariants of the orders in that column. Moreover, the order $\mathcal{O}$ in $\mathbb{H}_\mathfrak{p}$ with $d(\mathcal{O}) = \mathfrak{p}$ is the maximal order and has $e(\mathcal{O}) = -1$, and the orders in $M_2(K)$ with $d(\mathcal{O}) = \mathfrak{p}$ have $e(\mathcal{O}) = 1$.

We now turn to the 2-adic case. The only relations for the non-diagonal forms are given by (5.6) and (5.7). For the diagonal forms, we have as in the non-dyadic case that

$$f_1 = \langle 1 \rangle \perp \langle \delta_1 \pi^{r_1} \rangle \perp \langle \epsilon_1 \pi^{s_1} \rangle \sim f_2 = \langle 1 \rangle \perp \langle \delta_2 \pi^{r_2} \rangle \perp \langle \epsilon_2 \pi^{s_2} \rangle,$$

implies that $r_1 = r_2$ and $s_1 = s_2$. Conversely, if $r_1 = r_2$, $s_1 = s_2$, $\delta_1 \equiv \delta_2 \pmod 8$ and $\epsilon_1 \equiv \epsilon_2 \pmod 8$, then $f_1 \sim f_2$. Hence, we have at

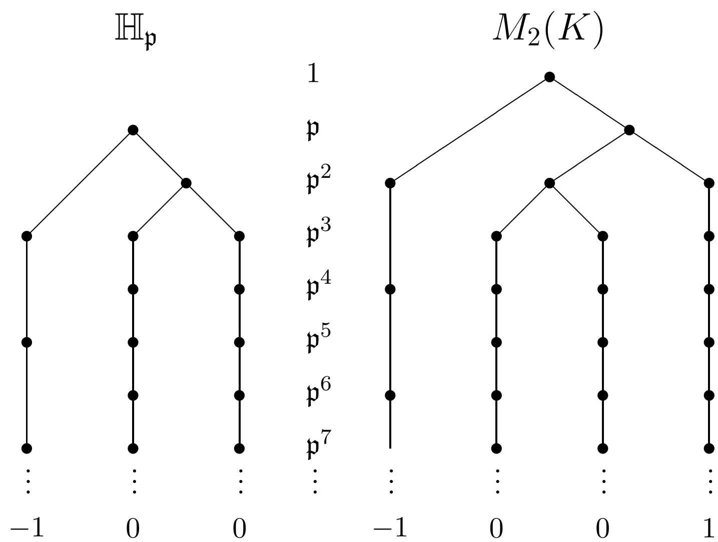

FIGURE 1. The trees of isomorphism classes of Bass orders in the non-dyadic case.

most 16 classes for every pair (r, s). But if $r < 3$ or $s - r < 3$, then the number of classes is reduced. We have used the 'canonical form' in [15] rather than [23] in order to get the following set of representatives:

(5.12) Proposition. *Let $\mathfrak{p} = (2)$ and let $\mathcal{O}$ be a Gorenstein order in a quaternion algebra over K. Then $\mathcal{O} \cong C_0(f)$, where f is uniquely chosen among the quadratic forms in Table 2.*

For example 11. in Table 2 gives rise to 4 classes for every $r \geq 4$, namely $(\delta_2, \delta_3) \in \{(5,1), (5,5), (7,1), (7,5)\}$. We get from (5.8), that the Bass orders are those in $\{1, 2, 5, 8, 9, 10, 11\}$. Now with (5.12) at our disposal, we can determine the number of isomorphism classes of orders also in the 2-adic case. We summarise everything in Table 3. The numbers n and β in Table 3 satisfy

$$n \geq 9 \text{ and } \beta = \begin{cases} \frac{1}{3}, & \text{if } n \equiv 2 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

The trees of isomorphism classes of Bass orders in the 2-adic case are shown in Figure 2.

6. Quaternion orders in the number field case

In this section, K will be an algebraic number field with ring of integers R.

1.	$H \perp \langle 2^r \rangle, r \geq 0$	12.	$\langle 1 \rangle \perp \langle \delta_4 2^2 \rangle \perp \langle \delta_4 2^r \rangle, r \geq 2$
2.	$J \perp \langle 2^r \rangle, r \geq 1$	13.	$\langle 1 \rangle \perp \langle 2^2 \rangle \perp \langle 3 \cdot 2^r \rangle, r \geq 2$
3.	$\langle 1 \rangle \perp 2^r H, r \geq 2$	14.	$\langle 1 \rangle \perp \langle 3 \cdot 2^2 \rangle \perp \langle 2^r \rangle, r \geq 3$
4.	$\langle 1 \rangle \perp 2^r J, r \geq 3$	15.	$\langle 1 \rangle \perp \langle \delta_7 2^2 \rangle \perp \langle 7 \cdot 2^r \rangle, r \geq 5$
5.	$\langle 1 \rangle \perp \langle 1 \rangle \perp \langle \delta_1 2^r \rangle, r \geq 0$	16.	$\langle 1 \rangle \perp \langle \delta_2 2^2 \rangle \perp \langle 2^r \rangle, r \geq 5$
6.	$\langle 1 \rangle \perp \langle 3 \rangle \perp \langle 2^r \rangle, r \geq 2$	17.	$\langle 1 \rangle \perp \langle 2^r \rangle \perp \langle \delta_5 2^{r+s} \rangle, r \geq 3, s \geq 0$
7.	$\langle 1 \rangle \perp \langle 7 \rangle \perp \langle 2^r \rangle, r \geq 3$	18.	$\langle 1 \rangle \perp \langle 7 \cdot 2^r \rangle \perp \langle \delta_6 2^{r+s} \rangle, r \geq 3, s \geq 0$
8.	$\langle 1 \rangle \perp \langle 5 \rangle \perp \langle \delta_1 2^r \rangle, r \geq 3$	19.	$\langle 1 \rangle \perp \langle 3 \cdot 2^r \rangle \perp \langle \delta_3 2^{r+s} \rangle, r \geq 3, s \geq 1$
9.	$\langle 1 \rangle \perp \langle 3 \cdot 2 \rangle \perp \langle \delta_1 2^r \rangle, r \geq 1$	20.	$\langle 1 \rangle \perp \langle 7 \cdot 2^r \rangle \perp \langle \delta_3 2^{r+s} \rangle, r \geq 3, s \geq 3$
10.	$\langle 1 \rangle \perp \langle 2 \rangle \perp \langle \delta_3 2^r \rangle, r \geq 3$	21.	$\langle 1 \rangle \perp \langle 3 \cdot 2^r \rangle \perp \langle \delta_6 2^{r+s} \rangle, r \geq 3, s \geq 3$
11.	$\langle 1 \rangle \perp \langle \delta_2 2 \rangle \perp \langle \delta_3 2^r \rangle, r \geq 4$	22.	$\langle 1 \rangle \perp \langle 5 \cdot 2^r \rangle \perp \langle \delta_5 2^{r+s} \rangle, r \geq 3, s \geq 3$

TABLE 2. Representatives of the similarity classes of primitive ternary quadratic forms over 2-adic integers. The δ_i:s are to be chosen arbitrarily according to $\delta_1 \in \{1,3\}$, $\delta_2 \in \{5,7\}$, $\delta_3 \in \{1,5\}$, $\delta_4 \in \{1,7\}$, $\delta_5 \in \{1,3,5,7\}$, $\delta_6 \in \{3,7\}$ and $\delta_7 \in \{3,5\}$.

	BASS		GORENSTEIN		TOTALLY	
Disc.	$\mathbb{H}_2$	M_2	$\mathbb{H}_2$	M_2	$\mathbb{H}_2$	M_2
1	0	1	0	1	0	1
2	1	1	1	1	1	1
2^2	1	3	1	3	1	3
2^3	2	2	2	2	2	3
2^4	2	4	2	6	3	7
2^5	4	4	5	5	6	8
2^6	4	6	6	11	8	14
2^7	7	7	10	10	13	17
2^8	6	8	10	18	16	26
$2^n, n$ odd	7	7	$4(n-5)$	$4(n-5)$	$\frac{1}{12}(7n^2-46n+135)+\beta$	$\frac{1}{4}(3n^2-22n+75)+3\beta$
$2^n, n$ even	6	8	$3(n-5)$	$5(n-5)$	$\frac{1}{12}(7n^2-52n+156)+\beta$	$\frac{1}{4}(3n^2-20n+68)+3\beta$

TABLE 3. The number of isomorphism classes of Bass, Gorenstein and arbitrary orders in $\mathbb{H}_\mathfrak{p}$ and $M_2 = M_2(K)$, when $\mathfrak{p} = (2)$.

A powerful tool when studying orders in quaternion algebras over algebraic number fields is to consider completions $\mathcal{O}_\mathfrak{p} = \mathcal{O} \otimes_R R_\mathfrak{p}$ with respect to the primes ideals in R. The basic result which makes this so useful is the following local-global correspondence [28, prop. III.5.1]:

(6.1) Proposition. *If Λ' is a lattice on $\mathfrak{A}$, then there is a bijection between $\{$lattices on $\mathfrak{A}\}$ and $\{(\Lambda_\mathfrak{p})_{\mathfrak{p}\in\Omega_f} : \Lambda_\mathfrak{p}$ lattice on $\mathfrak{A}_\mathfrak{p}$, $\Lambda_\mathfrak{p} = \Lambda'_\mathfrak{p}$ for almost all $\mathfrak{p}\}$ given by*

$$\Lambda \mapsto (\Lambda_\mathfrak{p}) \text{ and } (\Lambda_\mathfrak{p}) \mapsto \Lambda = \{x \in \mathfrak{A} : x \in \Lambda_\mathfrak{p},\ \forall \mathfrak{p} \in \Omega_f\}.$$

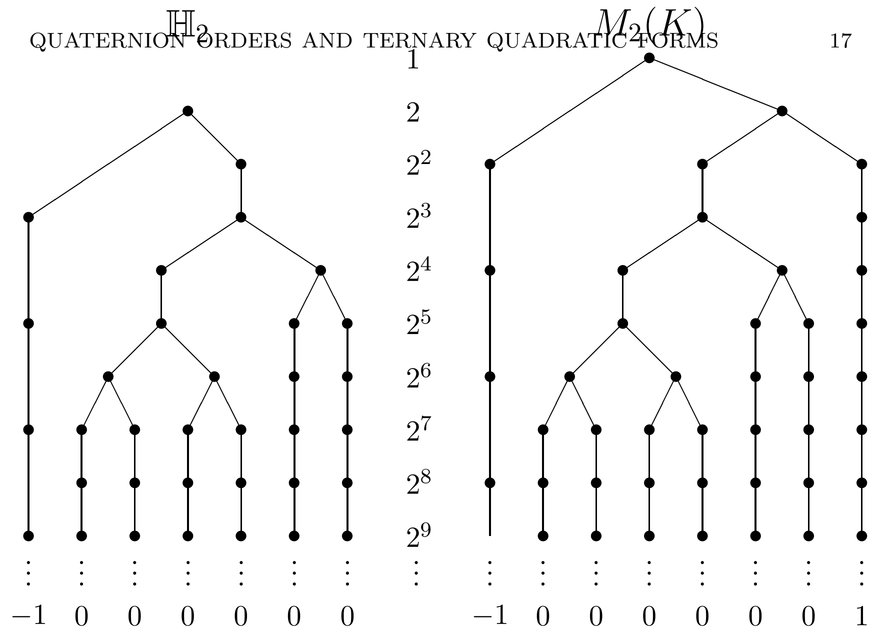

Figure 2. The trees of isomorphism classes of Bass orders in the 2-adic case.

The local-global principle remains true if we restrict to orders, since Λ is an order iff $\Lambda_{\mathfrak{p}}$ is an order for all $\mathfrak{p} \in \Omega_f$. It is immediate from the definitions that the discriminant and index satisfy

$$d(\mathcal{O})_{\mathfrak{p}} = d(\mathcal{O}_{\mathfrak{p}}) \text{ and } \left[\mathcal{O}'_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}}\right] = \left[\mathcal{O}' : \mathcal{O}\right]_{\mathfrak{p}}. \tag{6.2}$$

A direct consequence of (6.1) is that $\mathcal{O}$ is maximal iff $\mathcal{O}_{\mathfrak{p}}$ is maximal for all $\mathfrak{p} \in \Omega_f$. From this, (5.1) and (6.2), we get that

$$\mathcal{O} \text{ is maximal in } \mathfrak{A} \text{ iff } d(\mathcal{O}) = d(\mathfrak{A}). \tag{6.3}$$

It is also clear that $\mathcal{O}$ is a Bass (Gorenstein) order iff $\mathcal{O}_{\mathfrak{p}}$ is a Bass (Gorenstein) order for all $\mathfrak{p} \in \Omega_f$.

As already mentioned, the local-global principle does not apply to isomorphism classes since $\mathcal{O}_{\mathfrak{p}} \cong \mathcal{O}'_{\mathfrak{p}}\ \forall \mathfrak{p} \in \Omega_f$ does not imply $\mathcal{O} \cong \mathcal{O}'$ in general. We say that $\mathcal{O}$ and $\mathcal{O}'$ are in the same genus if $\mathcal{O}_{\mathfrak{p}} \cong \mathcal{O}'_{\mathfrak{p}}\ \forall \mathfrak{p} \in \Omega_f$, and that they are of the same type if $\mathcal{O} \cong \mathcal{O}'$. The number of types (isomorphism classes) in the genus of $\mathcal{O}$ is called the type number of $\mathcal{O}$ and will be denoted by $t(\mathcal{O})$.

Closely related to $t(\mathcal{O})$ are the one- and two-sided class numbers of $\mathcal{O}$, which are defined as follows. Let $L(\mathcal{O})$ be the set of locally principal left $\mathcal{O}$-ideals. We define an equivalence relation on $L(\mathcal{O})$ by $\Lambda_1 \sim \Lambda_2$ iff there exists $a \in \mathfrak{A}$ such that $\Lambda_1 = \Lambda_2 a$. The number of equivalence classes in $L(\mathcal{O})$ with respect to this relation is the one-sided class number $h(\mathcal{O})$ of $\mathcal{O}$. In the literature this is often simply referred to as the class number of $\mathcal{O}$. The facts that $h(\mathcal{O}) < \infty$ and that we get the same number of classes if we

take right ideals instead were first proved by Brandt in [1]. That paper is the first general investigation of the ideal theory of quaternion algebras.

If we restrict to two-sided ideals instead and do the same construction, then the number of equivalence classes is called the two-sided class number of $\mathcal{O}$ and will be denoted by $H(\mathcal{O})$.

(6.4) Remark. There exist orders with ideals which are not locally principal. For example, let $\mathfrak{p} = (\pi)$ be a principal ideal in R and

$$\mathcal{O} = (\pi, \pi)_R.$$

If $\Lambda = \{x \in \mathcal{O} : N(x) \in \mathfrak{p}\} = \mathfrak{p} + Ri + Rj + Rij$, then clearly Λ is a two-sided $\mathcal{O}$-ideal. Furthermore $[\mathcal{O} : \Lambda] = \mathfrak{p}$ and $\Lambda = \mathcal{O}i + \mathcal{O}j$.

If $g = \pi\alpha + \beta i + \gamma j + \delta ij \in \Lambda_{\mathfrak{p}}$, then

$$\begin{pmatrix} g \cdot 1 \\ g \cdot i \\ g \cdot j \\ g \cdot ij \end{pmatrix} = \begin{pmatrix} \pi\alpha & \beta & \gamma & \delta \\ \pi\beta & \pi\alpha & -\delta & -\gamma \\ \pi\gamma & \pi\delta & \pi\alpha & \beta \\ -\pi^2\delta & -\pi\gamma & \pi\beta & \pi\alpha \end{pmatrix} \begin{pmatrix} 1 \\ i \\ j \\ ij \end{pmatrix} =: A \begin{pmatrix} 1 \\ i \\ j \\ ij \end{pmatrix}.$$

Hence, $[\mathcal{O}_{\mathfrak{p}} : \mathcal{O}_{\mathfrak{p}} g] = (\det(A)) \subseteq \mathfrak{p}^2$ and this implies that $\Lambda_{\mathfrak{p}}$ is not principal since $[\mathcal{O}_{\mathfrak{p}} : \Lambda_{\mathfrak{p}}] = \mathfrak{p}$. □

Let $\Lambda_1, \ldots, \Lambda_{h(\mathcal{O})}$ be a set of representatives of left $\mathcal{O}$-ideal classes. Then every order in the genus of $\mathcal{O}$ is isomorphic to at least one of the right orders $\mathcal{O}_r(\Lambda_i)$. In fact, if $\mathcal{O}'$ is in the genus of $\mathcal{O}$, then $\mathcal{O}'$ is isomorphic to exactly $H(\mathcal{O}')$ of the orders $\mathcal{O}_r(\Lambda_i)$. If we fill in the details, we get a proof of the following proposition. For an adélic proof, see [28, p. 88].

(6.5) Proposition. *Let $\mathcal{O}$ be an arbitrary order in a quaternion algebra over K. Then*

$$h(\mathcal{O}) = \sum_{i=1}^{t(\mathcal{O})} H(\mathcal{O}_i),$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_{t(\mathcal{O})}$ are a set of representatives of the types in the genus of $\mathcal{O}$. In particular, $h(\mathcal{O}) = h(\mathcal{O}')$, if $\mathcal{O}$ and $\mathcal{O}'$ are in the same genus.

For results regarding the calculation of class and type numbers, see [4], [7], [10], [11], [18], [25], [26] and [SJ2].

(6.6) Remark. There is a little confusion on notions of 'class number' regarding quadratic forms versus quaternion orders. Let N be the norm form of an order $\mathcal{O}$. The class number of the quadratic form N is defined to be the number of isometry classes in the genus of N, that is, forms locally isometric to N. Hence, the class number of N is equal to the type number of $\mathcal{O}$. □

(6.7) Example. In order to illustrate our discussion, we will as an example determine the number of isomorphism classes of orders $\mathcal{O}$ in rational quaternion algebras with $d(\mathcal{O}) = (72) = (2^3 \cdot 3^2)$. If $\mathcal{O}$ is an order in an algebra $\mathfrak{A}$, then $d(\mathfrak{A})$ divides $d(\mathcal{O})$. Since the discriminant determines the

rational quaternion algebra and every positive square free integer is the discriminant of a quaternion algebra, we get 4 possibilities: $d(\mathfrak{A}) \in \{1, 2, 3, 6\}$. The cases $d(\mathfrak{A}) \in \{2, 3\}$ are definite algebras and the other two are indefinite.

From Table 1, we get that there are 1 isomorphism class of orders with $d(\mathcal{O}_3) = (72) = (3^2)$ in $\mathbb{H}_3$ and 3 in $M_2(\mathbb{Q}_3)$. Furthermore, Table 3 implies that there are 2 isomorphism classes of orders with $d(\mathcal{O}_2) = (72) = (2^3)$ in $\mathbb{H}_2$ and 3 in $M_2(\mathbb{Q}_2)$. Of all these orders only one in $M_2(\mathbb{Q}_2)$ is not Gorenstein or Bass. Now from (6.1), we derive the second column of Table 4.

$d(\mathfrak{A})$	#genera	#types
1	9	9
2	6	10
3	3	3
6	2	2

TABLE 4. The number of genera and types of rational orders with discriminant equal to (72).

It is easy to show, using for example the results in [7], that the type numbers of the orders in the two indefinite algebras are all equal to 1 in our case. The easiest way to determine the type numbers of the definite orders is to check the tables in [3]. These tables reveal that 4 of the genera in the algebra ramified at 2 have 2 classes and all other genera only one. This gives the last column of Table 4. Of course, the tables in [3] give all this information concerning the definite algebras in this case. □

We conclude by giving an explicit basis of a maximal order in an arbitrary quaternion algebra $\mathfrak{A}$ with $d(\mathfrak{A}) = (d)$ principal. First choose a generator a of a prime ideal satisfying (2.11), so that $\mathfrak{A} \cong (a, -d)_K$. We start with the order $\mathcal{O}' = (a, -d)_R$, which has $d(\mathcal{O}') = (4ad)$. From (3.2) and (6.3), we get that an order $\mathcal{O} \supseteq \mathcal{O}'$ is maximal iff $[\mathcal{O} : \mathcal{O}'] = (4a)$. The last condition in (2.11) implies that

$$\exists x \in R : a \equiv x^2 \pmod 4.$$

We have $(\frac{-d}{a}) = 1$, since $\mathfrak{A}$ is not ramified at a, and hence

$$\exists m \in R : -d \equiv m^2 \pmod a.$$

Now, if

$$e_1 = \frac{x+i}{2} \text{ and } e_2 = \frac{mi + ij}{a}, \tag{6.8}$$

then the norms and traces of e_1, e_2 and their products belong to R, since

$$N(e_1) = \tfrac{x^2-a}{4},\ Tr(e_1) = x,$$
$$N(e_2) = -\tfrac{d+m^2}{a},\ Tr(e_2) = 0 \text{ and } Tr(e_1e_2) = m.$$

Hence, we get that $\mathcal{O} = \langle 1, e_1, e_2, e_1e_2 \rangle$ is an order. The matrix which takes $1, i, j, ij$ to $1, e_1, e_2, e_1e_2$ has determinant equal to $\frac{1}{4a}$. Hence, $[\mathcal{O} : \mathcal{O}'] = (4a)$ and we have proved the following:

(6.9) Proposition. *Let $\mathfrak{A}$ be a quaternion algebra over K with $d(\mathfrak{A}) = (d)$ principal. Let a be chosen according to (2.11), so that $\mathfrak{A} \cong (a, -d)_K$. Then $\mathcal{O} = \langle 1, e_1, e_2, e_1e_2 \rangle$ is a maximal order in $\mathfrak{A}$, where e_1 and e_2 are defined by (6.8).*

We remind that in general there may be other non-isomorphic maximal orders in $\mathfrak{A}$. However, if $\mathfrak{A}$ is for example a totally indefinite quaternion algebra over a field K with class number of K equal to 1, then the maximal order is unique up to isomorphism [10].

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